Madey's and Liouville's theorems relating to free-electron lasers without inversion

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The small-gain regime of free-electron lasers (FEL) without inversion is considered and seeming contradictions with the traditional theory of FEL are resolved. As a result, a generalized Madey's theorem is obtained for the case of a phase shift given to electrons between the two wigglers. It explicitly demonstrates the contribution of interference of radiation from the wigglers. It is shown, by considering the motion in the phase space, that Liouville's theorem applied to the motion of electrons in a two-dimensional (rather than one-dimensional) real space is consistent with a nonzero integral of the gain over electron energies. These conclusions are crucial for the achievement of orders-of-magnitude improvement in the FEL gain in the case of a large electron energy spread. [S1063-651X(98)03203-6]

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I. INTRODUCTION

Free-electron lasers (FEL) [1] use the kinetic energy of relativistic electrons moving through a spatially modulated magnetic field (wiggler) to produce coherent radiation. The frequency of radiation is determined by the energy of electrons and the spatial period of the magnetic field. This permits tuning a FEL in a broad range unlike atomic or molecular lasers. Therefore for the purposes of achievement of short-wavelength lasers, it is important to consider possible limitations of the FEL gain.

For the case of a small-gain and a small-signal regime (i.e., laser intensity is much less than the saturation intensity) Madey has proved a theorem [2], which greatly simplifies the calculation of the FEL gain. Some generalizations of this theorem have been found [3]. The statement of the theorem is that the FEL gain is proportional to the derivative of the power of spontaneous emission when both are considered as functions of the injected electron energy $\mathcal{E}_i$,

$$\text{gain} \propto \frac{d}{d\mathcal{E}_i}\text{spontaneous emission}. \tag{1}$$

This immediately imposes the restriction that

$$\int_{-\infty}^{+\infty} \text{gain} \;\; d\mathcal{E}_i = 0, \tag{2}$$

since spontaneous emission turns to zero both at very high and very low injected energies.

Liouville's theorem [4], which is valid for Hamiltonian dynamical systems, states that the density of noninteracting particles in the one-particle phase space is conserved along their trajectories. This leads to the same conclusion (2) of a vanishing gain integral in the case of an effectively one-dimensional motion of the electrons [5]. This property of the FEL gain leads in the presence of a wide energy spread of electrons to a fast decrease (faster than in the case of Doppler broadening in atomic lasers) of the gain. Since the relative width of gain as a function of the electron energy decreases with the laser wavelength, it is one of the major obstacles on the way to short-wavelength FELs.

Recently new approaches to increasing gain in atomic lasers based on quantum coherence and interference have been proposed [6]. This concept, lasing without inversion, has interesting implications for the FEL as well, leading to a new type of FEL [7], the free-electron laser without inversion (FELWI). The FELWI is conceptually implemented via interference of the radiation from two wigglers and an appropriate phasing of the electrons in the static magnetic field of the drift region between the wigglers. The static magnetic field deflects the electrons to angles depending on their velocities so that the electrons travel along different paths and encounter different phases of the laser field in the second wiggler. Electrons having smaller velocities than the phase velocity of the ponderomotive potential are given an additional phase shift of π via inhomogeneity of magnetic field. As a result, the integral of the gain is nonzero. This promises a much higher gain for short-wavelength FELs, but at the same time seems to contradict some well-established notions of the FEL theory. The aim of this paper is to resolve these contradictions by explicitly demonstrating how modified versions of the conventional reasoning are applicable to FELWI, and thereby to support the possibility of a practical implementation of FELWI schemes.

In Sec. II the equations of motion for FELs are derived for further reference. In Sec. III the modified Madey theorem is illustrated for the case of uniform wigglers, and the numerical as well as analytical results for FELWI are discussed. The theorem is proved for the general case in Appendix A. Section IV shows how a consistent application of Liouville's theorem allows us to avoid contradictions with the FELWI results. And finally, the ways to implement the phase delay of electrons in the drift region between the wigglers in order to achieve necessary interference are discussed in Appendix B.

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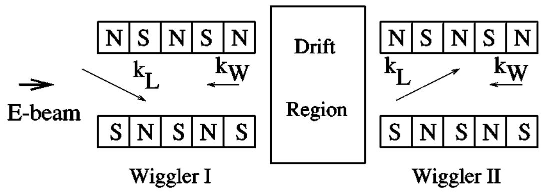

FIG. 1. The setup of FELWI: two wigglers and a drift region between them, $\mathbf{k}_L$ and $\mathbf{k}_W$ are the wave vectors for laser field and wiggler field, respectively.

II. EQUATIONS OF MOTION FOR FEL

The classical dynamics of electrons in the FEL can be represented (we disregard the scalar potential) by the Hamiltonian [8]

$$H=\mathcal{E}\equiv \gamma mc^2=c\sqrt{(\mathbf{p}-e\mathbf{A})^2+m^2c^2}, \tag{3}$$

where m and e are the mass and the charge of the electron, c is the speed of light, γ is referred to as the Lorenz factor related to the energy $\mathcal{E}$, $\mathbf{p}$ is the canonical momentum, and $\mathbf{A}$ is the vector potential of the combined electromagnetic field of the wiggler (designated by a subscript W) which is oriented along the z axis, and the laser field (designated by a subscript L) which propagates at an angle θ relative to the axis of the wiggler, as is shown in Fig. 1. For the uniform wiggler the vector potential has the form

$$\mathbf{A}=2\hat{\mathbf{y}}[A_W\cos(k_W z)+A_L\cos(k_z z+k_x x-\nu t+\phi)], \tag{4}$$

where $ck_z=\nu\cos\theta$ and $ck_x=\nu\sin\theta$. Both fields have only a y component, ϕ is the phase of the laser field at the moment when the electron enters the wiggler. Let us remark that we have neglected the spatial variation of the laser field (as we restrict our consideration to the small-gain regime) similarly to [1].

Since the interaction in a FEL happens within fixed spatial coordinates, it is more convenient to take the z coordinate rather than t as the independent variable. This leads to a canonical transformation (see [8] for details) with the new Hamiltonian $-\mathcal{K}$,

$$\mathcal{K}=p_z=eA_z+\sqrt{\frac{\mathcal{E}^2}{c^2}-(\mathbf{p}_\perp-e\mathbf{A}_\perp)^2-m^2c^2}, \tag{5}$$

where $\mathbf{p}_\perp$ and $\mathbf{A}_\perp$ are the transverse momentum and the transverse vector potential, respectively. Now time t and energy $\mathcal{E}$ play the role of a conjugate coordinate and momentum. Thus the equations of motion are as follows:

$$\frac{dp_y}{dz}=\frac{\partial\mathcal{K}}{\partial y}=0, \tag{6}$$

$$\frac{dp_x}{dz}=\frac{\partial\mathcal{K}}{\partial x}=-\frac{e^2}{p_z}\frac{\partial}{\partial x}\left(\frac{A_y^2}{2}\right), \tag{7}$$

$$\frac{d\mathcal{E}}{dz}=-\frac{\partial\mathcal{K}}{\partial t}=\frac{e^2}{p_z}\frac{\partial}{\partial t}\left(\frac{A_y^2}{2}\right), \tag{8}$$

$$\frac{dy}{dz}=-\frac{\partial\mathcal{K}}{\partial p_y}=\frac{-eA_y}{p_z}, \tag{9}$$

$$\frac{dx}{dz}=-\frac{\partial\mathcal{K}}{\partial p_x}=\frac{p_x}{p_z}, \tag{10}$$

$$\frac{dt}{dz}=\frac{\partial\mathcal{K}}{\partial\mathcal{E}}=\frac{\mathcal{E}}{p_z c^2}. \tag{11}$$

Here we take $p_y(0)=0$ and therefore $p_y(z)=0$ for all z. This allows us from now on to take the wiggler motion in the y direction out of consideration for the purposes of calculating gain.

The right side of Eq. (7) contains, among other terms, the ''ponderomotive'' (wiggler plus radiation) field which propagates along the z direction with the phase velocity $v_r=\nu/q_z<c$, where $q_z=k_z+k_W$. The electrons having velocities close to the resonant velocity v_r interact with this field most efficiently. We retain only these terms (an analog of the rotating wave approximation, see [9]) and neglect the other terms having fast oscillation in the electron reference frame. This results in the proportionality of the energy change to the change of momentum for each electron in each act of photon absorption or emission:

$$\frac{d\mathcal{E}}{dz}=\mathcal{N}\sin(q_z z+k_x x-\nu t+\phi), \tag{12}$$

$$\mathcal{N}=\frac{2e^2\nu A_W A_L}{p_z}, \tag{13}$$

$$\frac{dp_x}{dz}=\frac{k_x}{\nu}\frac{d\mathcal{E}}{dz}, \tag{14}$$

$$\frac{dp_z}{dz}=\frac{q_z}{\nu}\frac{d\mathcal{E}}{dz}. \tag{15}$$

Let us introduce the phase ψ relative to the ponderomotive potential as

$$\psi=q_z z+k_x x-\nu t+\phi, \tag{16}$$

and let us call ''detuning'' the z derivative of ψ

$$\Omega=q_z+k_x\frac{dx}{dz}-\nu\frac{dt}{dz}, \tag{17}$$

since at $z=0$ it is proportional to the difference of the electron velocity from the resonant velocity v_r. Finally, with the help of the equations of motion (7)–(11), the second derivatives of the coordinates can be expressed via the derivative of energy (p_x, being of the first order in the laser field, is neglected in comparison with p_z)

$$\frac{d^2t}{dz^2}=\frac{1}{p_z c^2}\left(1-\frac{\mathcal{E}q_z}{\nu p_z}\right)\frac{d\mathcal{E}}{dz}, \tag{18}$$

$$\frac{d^2x}{dz^2}=\frac{k_x}{p_z\nu}\frac{d\mathcal{E}}{dz}. \tag{19}$$

Thus we obtain the equation

$$\frac{d\Omega}{dz}=\frac{1}{p_z\nu}\left(k_x^2+\frac{q_z^2}{\gamma_r^2}\right)\frac{d\mathcal{E}}{dz}. \tag{20}$$

Therefore in terms of variables (ψ,Ω) the dynamics of an electron is expressed by the pendulum equations

$$\frac{d\psi}{dz}=\Omega,$$

$$\frac{d\Omega}{dz}=a\ \sin\ \psi, \tag{21}$$

where the coupling constant

$$a=\left(k_x^2+\frac{q_z^2}{\gamma_r^2}\right)\frac{2e^2A_WA_L}{p_z^2}, \tag{22}$$

is proportional to the laser field amplitude and will be used as the perturbation parameter. Equations (21) are associated with the reduced (one-dimensional) Hamiltonian

$$\mathcal{H}(\psi,\Omega)=\frac{\Omega^2}{2}+a\ \cos\ \psi, \tag{23}$$

which is the nonrelativistic Hamiltonian of an electron in the reference frame of the ponderomotive potential [10]. The dynamics of electrons inside the wiggler is effectively one dimensional in spite of the two-dimensional motion (in the x-z plane) caused by the oblique propagation of laser field. These equations are the basis of our consideration of uniform wigglers. In the ultrarelativistic limit, small changes of the energy, momentum, detuning, and velocity are proportional to each other, so in order to calculate the gain we will need to calculate the change in the average detuning of the electrons.

III. SMALL-GAIN REGIME OF FELWI

Here we apply the equations derived in Sec. II to the electron motion in one and two uniform wigglers and demonstrate the traditional and our modified Madey's theorems. We will look for the solution of Eqs. (21) in the form of a perturbation series in the amplitude of the laser field:

$$\Omega(z)=\Omega_0(z)+a\Omega_1(z)+a^2\Omega_2(z)+\cdots, \tag{24}$$

$$\psi(z)=\psi_0(z)=a\psi_1(z)+\cdots, \tag{25}$$

where the initial conditions state that electrons start with the phase ϕ and the injected detuning Ω_i,

$$\Omega_0(0)=\Omega_i,\quad \psi_0(0)=\phi, \tag{26}$$

$$\Omega_n(0)=0,\quad \psi_n(0)=0,\quad n>0. \tag{27}$$

The set of the equations becomes

$$\frac{d}{dz}\Omega_0=0, \tag{28}$$

$$\frac{d}{dz}\psi_0=\Omega_0, \tag{29}$$

$$\frac{d}{dz}\Omega_1=\sin\ \psi_0, \tag{30}$$

$$\frac{d}{dz}\psi_1=\Omega_1, \tag{31}$$

$$\frac{d}{dz}\Omega_2=\psi_1\ \cos\ \psi_0. \tag{32}$$

Consider first a one-section wiggler of length L_W. Integrating Eqs. (28)–(32), we directly obtain

$$\Omega_1=\frac{2\ \sin(\Omega_i z/2)\sin[(\Omega_i z/2)+\phi]}{\Omega_i}, \tag{33}$$

$$\psi_1=\frac{\Omega_i z\ \cos(\phi)+\sin\ \phi-\sin(\Omega_i z+\phi)}{\Omega_i^2}, \tag{34}$$

$$\Omega_2=\frac{-4-\cos 2\phi+4\cos\ \Omega_i z-\cos 2(\Omega_i z+\phi)+2\Omega_i z\ \sin\ \Omega_i z+2\Omega_i z\ \sin(2\phi+\Omega_i z)}{4\Omega_i^3}. \tag{35}$$

Electrons are injected with random phases. Averaging over the phase ϕ cancels the contribution to the gain in the first order. In the second order it yields

$$\text{gain}\propto-\langle\Omega_2\rangle=\frac{2-2\ \cos(\Omega_i L_W)-\Omega_i L_W\ \sin(\Omega_i L_W)}{2\Omega_i^3}. \tag{36}$$

It can be shown [8] that the spontaneous emission power from the electrons is proportional to the square of the energy change (which we take in the first order of perturbation)

$$\text{spontaneous emission}\propto W\equiv\langle\Omega_1^2\rangle. \tag{37}$$

The above solution gives for the one-section wiggler

$$W_I(\Omega_i)=\left(\frac{\sin(\Omega_i L_W/2)}{\Omega_i}\right)^2. \tag{38}$$

The same result can be obtained from the quantum theory of FEL. Comparing Eq. (36) and Eq. (38), we explicitly see a manifestation of Madey's theorem:

$$\text{gain} \propto -\langle \Omega_2 \rangle = -\frac{d}{d\Omega_i} \langle \Omega_1^2(\Omega_i) \rangle, \tag{39}$$

in other words, the gain is proportional to the derivative of the spontaneous emission spectrum. Let us suppose now that we have a two-section wiggler, with the second section being identical to the first. In the drift region between the two wigglers the electrons can be given a velocity-dependent shift $\Delta\phi$ of their phase relative to the ponderomotive potential. To achieve a positive integral of the gain over detuning [11] we design the drift region so that the phase shift depends on the electrons' injected velocity, and consequently on Ω_i.

A calculation similar to the one above gives for the corresponding values at the exit from the second wiggler

$$\Omega_1 = \frac{\cos(\Omega_i L_W + \phi + \Delta\phi) - \cos(\phi + \Omega_i L_W) + \cos\phi - \cos(\Omega_i z + \phi + \Delta\phi)}{\Omega_i},$$

$$\psi_1 = [\Omega_i z \cos\phi + \Omega_i L_W \cos(\phi + \Omega_i L_W) - \Omega_i z \cos(\phi + \Omega_i L_W) - \Omega_i L_W \cos(\Delta\phi + \phi + \Omega_i L_W) + \Omega_i z \cos(\Delta\phi + \phi + \Omega_i L_W)$$
$$+ \sin\phi - \sin(\phi + \Omega_i L_W) + \sin(\Delta\phi + \phi + \Omega_i L_W) - \sin(\Delta\phi + \phi + \Omega_i z)]/\Omega_i^2,$$

$$\Omega_2 = \{-8 + 4\cos\Delta\phi - \cos 2\phi + 4\cos\Omega_i L_W + 4\cos\Omega_i(L_W - z) - 4\cos(\Delta\phi + \Omega_i L_W) + \cos 2(\phi + \Omega_i L_W)$$
$$- \cos 2(\Delta\phi + \phi + \Omega_i L_W) + 4\cos(\Delta\phi + \Omega_i z) + \cos 2(\Delta\phi + \phi + \Omega_i z) - 4\cos[\Delta\phi + \Omega_i L_W(z - L_W)] + 2\Omega_i L_W \sin(\Omega_i L_W)$$
$$+ 2\Omega_i L_W \sin\Omega_i(L_W - z) - 2\Omega_i z \sin\Omega_i(L_W - z) - 2\Omega_i L_W \sin(\Delta\phi + \Omega_i L_W)$$
$$+ 2\Omega_i L_W \sin(2\phi + \Omega_i L_W) - 2\Omega_i L_W \sin(\Delta\phi + 2\phi + \Omega_i L_W) + 2\Omega_i z \sin(\Delta\phi + \Omega_i z) + 2\Omega_i z \sin(\Delta\phi + 2\phi + \Omega_i z)$$
$$+ 2\Omega_i L_W \sin[\Delta\phi + \Omega_i(z - L_W)] - 2\Omega_i z \sin[\Delta\phi + \Omega_i(z - L_W)] + 2\Omega_i L_W \sin[\Delta\phi + 2\phi + \Omega_i(L_W + z)]$$
$$- 2\Omega_i z \sin[\Delta\phi + 2\phi + \Omega_i(L_W + z)] - 2\Omega_i L_W \sin[2\Delta\phi + 2\phi + \Omega_i(L_W + z)] + 2\Omega_i z \sin[2\Delta\phi + 2\phi + \Omega_i(L_W$$
$$+ z)]\}/(4\Omega_i^3).$$

Averaging over the initial phase yields

$$\text{gain} \propto -\langle \Omega_2 \rangle = -\{-4 + 2\cos\Delta\phi + 2\cos\Omega_i L_W + 2\cos\Omega_i(L_W - z) - 2\cos(\Delta\phi + \Omega_i L_W) + 2\cos(\Delta\phi + \Omega_i z)$$
$$- 2\cos[\Delta\phi + \Omega_i(z - L_W)] + \Omega_i L_W \sin(\Omega_i L_W) + \Omega_i L_W \sin\Omega_i(L_W - z) - \Omega_i z \sin\Omega_i(L_W - z)$$
$$- \Omega_i L_W \sin(\Delta\phi + \Omega_i L_W) + \Omega_i z \sin(\Delta\phi + \Omega_i z) + \Omega_i L_W \sin[\Delta\phi + \Omega_i(z - L_W)]$$
$$+ \Omega_i z \sin[\Delta\phi + \Omega_i(z - L_W)]\}/(2\Omega_i^3). \tag{40}$$

Obviously, for zero $\Delta\phi$ the two-wiggler gain coincides with the result for the one-section wiggler of twice the length.

Spontaneous emission can be most readily estimated as the intensity of the superposition of radiation form the first (E_1) and the second (E_2) wiggler. Each wiggler separately gives $W_I \sim |E_{1,2}|^2$, and together

$$W_{II} \sim |E_1|^2 + |E_2|^2 + 2|E_1||E_2|\cos\varphi, \tag{41}$$

where

$$\varphi = \Omega_i T_W + \Delta\phi \tag{42}$$

is the difference between the phase of the same electron in the two wigglers. Thus

$$W_{II}(\Omega_i, \Delta\phi) = 4 W_I(\Omega_i) \cos^2 \frac{\varphi}{2}. \tag{43}$$

The same result can be obtained by directly evaluating

$$W_{II}(\Omega_i, \Delta\phi) = \langle \Omega_1^2(\Omega_i, \Delta\phi) \rangle. \tag{44}$$

Again by comparing Eq. (40) and Eq. (43) we explicitly see the relation

$$\text{gain} \propto -\langle \Omega_2 \rangle = -\frac{\partial}{\partial\Omega_i} \langle \Omega_1^2(\Omega_i, \Delta\phi) \rangle, \tag{45}$$

where the partial derivative means that the differentiation does not affect $\Delta\phi$ even though it depends on the initial detuning. We prove this modified Madey theorem for a more general class of interaction Hamiltonians.

The modified Madey theorem can also be expressed in the equivalent form

$$\text{gain} \propto -\langle \Omega_2 \rangle = -\frac{1}{2} \frac{d}{d\Omega_i} \langle \Omega_1^2 \rangle + \left\langle \Omega_1(L_W) \frac{\partial}{\partial\psi_i} \Omega_1(2L_W) \right\rangle \frac{\partial \Delta\phi}{\partial\Omega_i}. \tag{46}$$

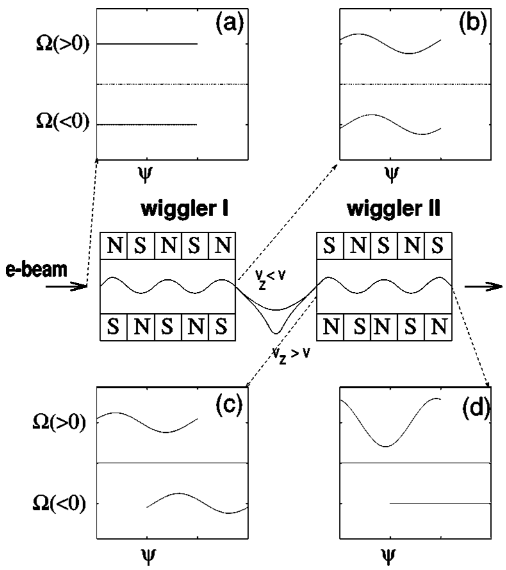

FIG. 2. Snapshots of the phase-space motion (Ω,ψ); (a) initial distribution of electrons before entering wigglers; (b) after the first wiggler; (c) after the drift region; (d) after the second wiggler.

Here, the first term represents the usual contribution to the gain, whereas the second term is a contribution due to interference. This relation can be regarded as a modification of Madey's theorem to include interference. The integral of gain for a one-section wiggler (39) over detunings has to be zero, since it contains only a full derivative. However, the integral of gain for a two-section wiggler (45) can be nonzero, if the phase shift $\Delta\phi$ in the interference term is properly chosen. This crucial difference opens new possibilities for building a new type of free-electron lasers having larger gain and being more tolerant to spread of electrons.

To demonstrate such a possibility we set

$$\Delta\phi=\begin{cases}\pi-\Omega L_W, & \Omega_i<0\\ -\Omega L_W, & \Omega_i>0\end{cases} \tag{47}$$

and find that the gain vanishes exactly for $\Omega_i<0$, and it is positive almost everywhere for $\Omega_i>0$. This is accompanied by a cancellation of spontaneous emissions for negative detunings.

Considering electron trajectories in the phase space of the pendulum, i.e., the plane $(\psi,\,\Omega)$, can help us obtain some physical insight into the previous results. In Figs. 2(a)–2(d) we present a number of snapshots of the evolution in phase space of a monoenergetic electron beam traveling through the FELWI. In Fig. 2(a), we see electrons entering the first wiggler with various phases and the same detuning. In the first wiggler the electron dynamics depends on their initial phases: some of the electrons lose energy but the others gain energy [see Fig. 2(b)]. Then the electrons enter the drift region where, traveling in the static magnetic field, they change their phases relative to the ponderomotive potential but do not change their energy (and thus their detuning). The change of phase is expressed by Eq. (47) and is determined by their *initial detuning* rather than the detuning at the exit from the first wiggler. The electrons with negative detuning reverse their motion inside the second wiggler. It is clearly seen from Fig. 2(d) that for negative detuning we have practically a monoenergetic electron beam exiting the FELWI. As a result, the gain for negative detunings is zero. For positive detunings, the evolution in the second wiggler is similar to that in the first wiggler: the electrons continue to lose (or gain) energy in the second wiggler, if they were losing (respectively, gaining) energy in the first. This results in a certain gain for the laser field.

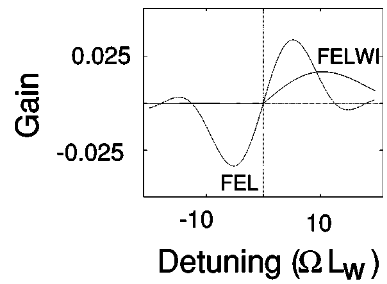

FIG. 3. Gain versus detuning for FEL and FELWI.

In Fig. 3 we compare the gain curves for an ordinary FEL with that of a FELWI. Clearly, the integral of the FELWI gain over all detunings is positive, in contrast to an ordinary FEL having a zero integral. In an ordinary FEL the majority of the electrons must have positive detunings to provide a net gain, which can be defined as inversion. FELWI does not have to satisfy this condition, which gives another justification for this term.

We emphasize the importance of the drift region design to produce the phase shift which depends on the detuning at the entrance to first wiggler rather than at the exit from the first wiggler. Phase trajectories demonstrate this most clearly. Suppose that instead of the phase shift (47) we have a different one,

$$\Delta\phi=\begin{cases}\pi-\Omega L_W, & \Omega<0\\ -\Omega L_W, & \Omega>0\end{cases} \tag{48}$$

where Ω stands for the electron detuning at the exit from the first wiggler (rather than Ω_i at the entrance to the first wiggler).

Then there is a region near zero detuning [12] where the phase shift causes a qualitative change in the phase-space distribution. Figures 4(a)–4(d) demonstrate this case for $\Omega_i=0$. In previous cases the contribution of the electrons losing energy was, to the first order of perturbation, canceled by the contribution of the electrons gaining energy, and the net gain or loss depended on the violation of this balance in the second order of perturbation. In the case of Eq. (48) this cancellation still exists in the first wiggler, but in the second wiggler all the electrons gain energy, so some loss exists even in the first order of perturbation. This explains the sharp absorption dip in Fig. 5 near zero detuning. We would like to note here that this result is in agreement with [12], where one-dimensional motion has also been analyzed and it is shown that FELWI is not possible for this case. We argue in the next section that an essentially two-dimensional motion of electrons between the wiggler is required to implement the phase shift (47).

The other result of our consideration of electron trajectories is that in the last case an extraordinarily high bunching of electrons takes place for the phase shift (48). This fact is

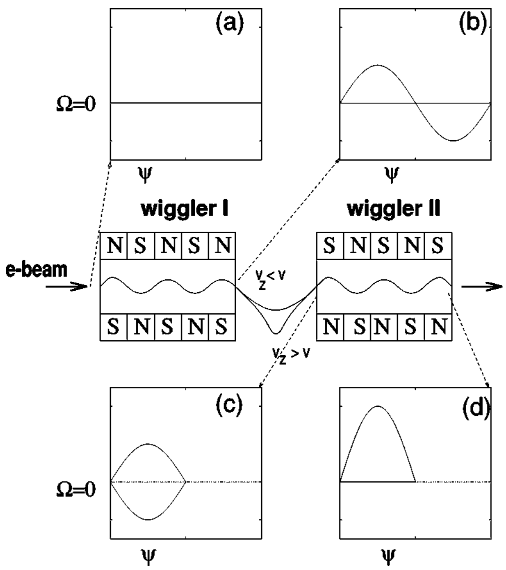

FIG. 4. Snapshots of the phase-space motion (Ω,ψ), for $\Omega_i=0$ and phase shift in the form (48); (a) initial distribution of electrons before entering wigglers; (b) after the first wiggler; (c) after the drift region; (d) after the second wiggler.

very important for the dynamics of the laser field in the large-gain regime. However, this is beyond the scope of the present paper and we plan to consider it in the future [13].

The integral gain can be increased even further by varying the parameters of the phase-shift function $\Delta\phi$ for the drift region:

$$\Delta\phi=\begin{cases}\alpha+\beta-\gamma\Omega L_W, & \Omega_i<0\\ \beta-\gamma\Omega L_W, & \Omega_i>0\end{cases}. \tag{49}$$

By adjusting the α, β, γ parameters it is possible not only to cancel absorption, but also to create gain for negative detunings. The peak gain higher than that of a usual FEL of the same total length can be achieved in a FELWI by wigglers with different lengths. But even without this optimization we are able to compare our FELWI gain with that experimentally obtained in [14]. Using the parameters of [14] (energy of electrons is 147 ± 4 MeV, energy spread 6.5×10^{-4}, transverse dimensions 0.26 and 0.15 mm, magnetic field amplitude 4.0 kG, magnetic period 4 cm, number of periods 23) we calculate the peak gain for different wavelengths, see Table I. The peak gains of FEL and FELWI are of the same order of magnitude because the inhomogeneous broadening is relatively small, while the average gain of

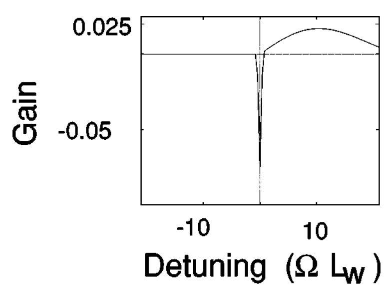

FIG. 5. Gain versus detuning, the large dip arises due to huge bunching.

TABLE I. Peak gain for different wavelengths.

Wavelength	488 nm	244 nm	122 nm	61 nm
Ratio of the FELWI gain to the FEL gain	0.78	1.0	1.6	2.5

FELWI is an order of magnitude larger. If we now compare the peak gain for shorter wavelengths using the same wiggler length, we find that FELWI demonstrates higher gain than an ordinary FEL.

IV. FREE-ELECTRON LASERS WITHOUT INVERSION AND LIOUVILLE'S THEOREM

Liouville's theorem [4] applies to a Hamiltonian system such as a single particle in external electric and magnetic fields or a collection of such particles with the number of particles conserved. If we consider an ensemble of such systems in the phase space of one such system, then the density of the systems is preserved along any trajectory of a system in the phase space. This theorem is applicable to individual particles in a beam when their mutual influence can be neglected; in particular, for an ordinary FEL it is valid in the small-gain regime. The theorem imposes some restrictions on the gain as a function of the electron energy. The aim of this part of our paper is to show that the existence of the FELWI does not violate Liouville's theorem. Moreover, considering the FELWI in terms of Liouville's theorem gives us some insight into how to design the drift region of a FELWI.

An important property of the FEL gain can be easily obtained by applying Liouville's theorem to the FEL as described by the pendulum equations (21). The mean value of detuning, and consequently of the energy, of electrons is given in terms of the phase density $f(\psi,\Omega,z)$ by

$$\langle\Omega\rangle=\int d\psi d\Omega \quad \Omega \quad f(\psi,\Omega,z). \tag{50}$$

Let us consider a rectangular region of the pendulum phase space, the phase being in the range $(0,2\pi)$ and the detuning in the range $(-\Omega_m,\Omega_m)$. Suppose the electrons are uniformly distributed in this rectangle at the entrance to the first wiggler with the distribution function

$$f(\psi,\Omega,0)=f_0, \tag{51}$$

as it is shown in Fig. 6(a). Note that in Fig. 6 we draw only a small but the most interesting part of the phase space located near the zero detuning. According to Liouville's theorem, the distribution function at any t equal to that at $z=0$,

$$f(\psi,\Omega,z)=f(\psi_i,\Omega_i,0)=f_0, \tag{52}$$

if the initial coordinates are within our rectangle. In the limit of $\Omega_m\to\infty$ all initial coordinates will fall into the rectangle. Since the distribution function remains constant at any z, so does the average energy $\langle\Omega\rangle$. Therefore the net gain for an infinitely wide flat distribution over the initial energies and, equivalently, the integral of gain for a monoenergetic distribution over the energy are equal to zero.

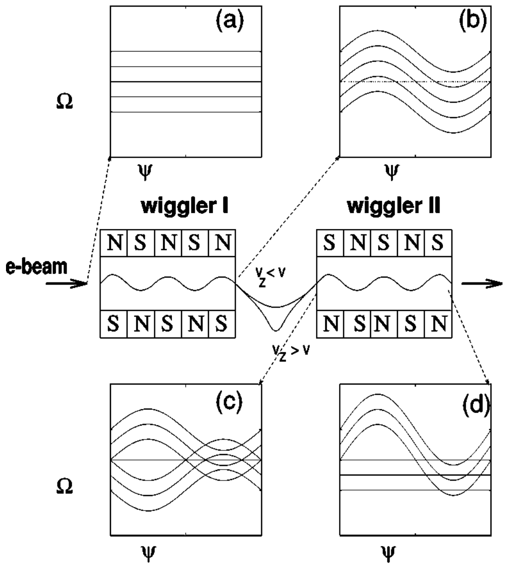

FIG. 6. Snapshots of the phase-space motion (Ω,ψ), curves correspond to the same initial detuning; (a) initial distribution of electrons before entering wigglers; (b) after the first wiggler; (c) after the drift region; (d) after the second wiggler; the intersection of curves means that the drift region in the form (47) changes the phase density.

The simplest way to overcome Liouville's restriction is to introduce a drift region between two wigglers. Motion inside the wigglers happens in the two-dimensional phase space (ψ,Ω), but inside the drift region the motion of the electrons is going on in the six-dimensional phase space (x,y,z,p_x,p_y,p_z). After the drift region, the electrons enter the second wiggler and acquire certain phases and detunings relative to the ponderomotive potential of the second wiggler. This amounts to the projection of the six-dimensional phase space (x,y,z,p_x,p_y,p_z) into a two-dimensional one (ψ,Ω). Even though the six-dimensional phase-space density is conserved, the two-dimensional projection does not have to be. These changes of the phase-space density give the possibility to have the gain for an electron beam with a large energy spread, by a proper design of the motion in the drift region.

In the following, using a simple example, we demonstrate that the projection of the phase density in the four-dimensional phase space (q,p,q_1,p_1), which is conserved, onto a two-dimensional space may not be conserved. Consider the model Hamiltonian

$$H=p_1 q, \tag{53}$$

where p_1, q are the canonical momentum and position coordinates, respectively, and t is the independent variable. There is no physical meaning to this simple Hamiltonian, but an equivalent type of Hamiltonian can arise in the case of electrons moving in a static magnetic field.

Our Hamiltonian leads to the set of equations

$$\dot{q}=0, \quad \dot{p}=-p_1, \quad \dot{q}_1=q, \quad \dot{p}_1=0, \tag{54}$$

the solutions of which are

$$q=q_0, \quad p=p_0-p_{10}t, \quad q_1=q_{10}+qt, \quad p_1=p_{10}, \tag{55}$$

where q_0, p_0, q_{10}, p_{10} label the initial position in the phase space.

We take a specific initial distribution

$$P(0,q_0,p_0,q_{10},p_{10})=\exp(-q_0^2-p_0^2-p_{10}^2-q_{10}^2) \tag{56}$$

as the density function at time $t=0$. One can express the initial coordinates in terms of the current ones with t as a parameter,

$$q_0=q, \quad p_0=p+p_1 t, \quad q_{10}=q_1-qt, \quad p_{10}=p_1. \tag{57}$$

Using the statement of Liouville's theorem that the phase-space density is preserved along trajectories

$$\frac{dP(t,q,p,q_1,p_1)}{dt}=0, \tag{58}$$

we can obtain the density function at any time

$$\begin{aligned} P(t,q,p,q_1,p_1)=\exp[&-q_0^2(t,q,p,q_1,p_1)\\ &-p_0^2(t,q,p,q_1,p_1)-p_{10}^2(t,q,p,q_1,p_1)\\ &-q_{10}^2(t,q,p,q_1,p_1)]. \end{aligned} \tag{59}$$

The density projected onto a two-dimensional subspace of the phase space is the following integral:

$$\rho(t,q,p)=\int\int dq_1 dp_1 P(t,q,p,q_1,p_1), \tag{60}$$

which amounts to

$$\rho(t,q,p)=\frac{\pi}{\sqrt{1+t^2}}\exp\left(-q^2-\frac{p^2}{1+t^2}\right). \tag{61}$$

Obviously it is not conserved with time along the trajectories.

Let us now show that the drift region producing the phase shift in the form (47) does change the density in the (Ω,ψ) phase space. At the exit from the first wiggler we will have basically the same rectangular region occupied by electrons but slightly distorted as it is shown in Fig. 6(b). The drift-region phase shift (47) leads to either an increase or a decrease of the phase density at the entrance to the second wiggler, as it is shown in Fig. 6(c). Moreover, the sign of the integral gain is clearly seen from the change of the phase density. Near $\Omega=0$ there are two regions [see Figs. 6(c) and 6(d)]: one has a negative density increment and is moving up, the other has a positive density increment and is moving down. The net contribution to the change of the electron detuning is negative so the gain of the FELWI is positive.

Let us remark here that the phase shift in the form (48) preserves the phase density, as can be seen from Figs. 7(a)–7(d), and therefore the integral of the gain over the detuning equals zero, as it has been pointed out in [12].

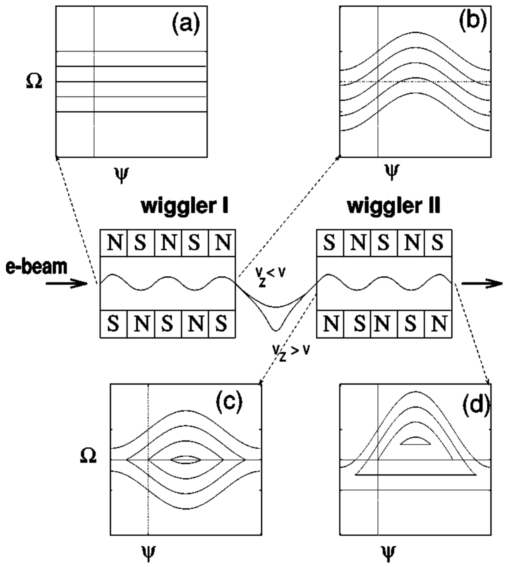

FIG. 7. Snapshots of the phase-space motion (Ω,ψ), curves correspond to the same initial detuning; (a) initial distribution of electrons before entering wigglers; (b) after the first wiggler; (c) after the drift region; (d) after the second wiggler; the phase density is preserved by the phase shift in the form (48).

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APPENDIX A: GENERALIZED MADEY'S THEOREM FOR FELWI

Here we prove in a very general case the modification of Madey's theorem which has been demonstrated for the uniform wiggler in Eq. (45). The proof closely follows that in [8], but accounts for the phase shift $\Delta\phi$ in the drift region. We assume the Hamiltonian to have the form

$$H=H_0(\mathbf{r},\mathbf{p})+V(\mathbf{r},\mathbf{p},t,\Delta\phi). \tag{A1}$$

The phase shift happens at a fixed point and does not influence the evolution at other points. We assume that the perturbation V is periodic and gives zero average over a period.

First we solve the equations of motion for the unperturbed Hamiltonian H_0 and obtain the coordinates and the momenta depending on the initial conditions:

$$\mathbf{r}_0=\mathbf{r}_0(\mathbf{r}_i,\mathbf{p}_i,t),\quad \mathbf{p}_0=\mathbf{p}_0(\mathbf{r}_i,\mathbf{p}_i,t). \tag{A2}$$

Then we make a canonical change of variables as follows:

$$\mathbf{r}'=\mathbf{r}_0(\mathbf{r},\mathbf{p},t),\quad \mathbf{p}'=\mathbf{p}_0(\mathbf{r},\mathbf{p},t). \tag{A3}$$

In the new coordinates, the Hamiltonian reduces to only the perturbation part (we drop the primes for the rest of this section)

$$\frac{d\mathbf{r}}{dt}=\frac{\partial V}{\partial \mathbf{p}},\quad \frac{d\mathbf{p}}{dt}=-\frac{\partial V}{\partial \mathbf{r}}. \tag{A4}$$

The zeroth approximation is simply $\mathbf{r}_0=\mathbf{r}_i$ and $\mathbf{p}_0=\mathbf{p}_i$. In the first approximation we must take V on the zeroth-order trajectories

$$\frac{d\mathbf{r}_1}{dt}=\frac{\partial V(\mathbf{r}_i,\mathbf{p}_i,t,\Delta\phi)}{\partial \mathbf{p}_i},\quad \frac{d\mathbf{p}_1}{dt}=-\frac{\partial V(\mathbf{r}_i,\mathbf{p}_i,t,\Delta\phi)}{\partial \mathbf{r}_i}. \tag{A5}$$

These derivatives do not apply to $\Delta\phi$ even though it might depend on the initial coordinates via the current coordinates. Hence any function of the dynamical variables can be calculated in this order of perturbation as

$$F(\mathbf{r},\mathbf{p},t)=F(\mathbf{r}_i,\mathbf{p}_i,t)+\frac{\partial F}{\partial \mathbf{r}_i}\mathbf{r}_1+\frac{\partial F}{\partial \mathbf{p}_i}\mathbf{p}_1. \tag{A6}$$

Second, we apply the above formalism to a still rather general case with z as the independent variable and the change of the Hamiltonian $H\to-\mathcal{K}$, $V\to-\mathcal{V}$. We keep the notation $\mathbf{r}$ and $\mathbf{p}$ for the transverse coordinate and momentum only. Let the total length of the wiggler be L. The solutions for the Hamilton equations in the first order are

$$\mathbf{r}_1=-\int_0^z\frac{\partial\mathcal{V}(\zeta)}{\partial \mathbf{p}_i}d\zeta, \tag{A7}$$

$$\mathbf{p}_1=\int_0^z\frac{\partial\mathcal{V}(\zeta)}{\partial \mathbf{r}_i}d\zeta, \tag{A8}$$

$$t_1=\int_0^z\frac{\partial\mathcal{V}(\zeta)}{\partial \mathcal{E}_i}d\zeta, \tag{A9}$$

$$\mathcal{E}_1=-\int_0^z\frac{\partial\mathcal{V}(\zeta)}{\partial t_i}d\zeta, \tag{A10}$$

where $\mathcal{V}(\zeta)$ designates $\mathcal{V}(\mathbf{r}_i,\mathbf{p}_i,t_i,\mathcal{E}_i,\zeta,\Delta\phi)$ and differentiation does not apply to the implicit dependence in $\Delta\phi$. We estimate the work $\mathcal{A}$ done on the electron,

$$\mathcal{A}=-\int_0^L\frac{\partial\mathcal{V}(z)}{\partial t}dz. \tag{A11}$$

Averaged over the phases, it is proportional to the negative of the gain. Substituting $\mathcal{A}$ into Eq. (A6) we arrive at

$$\mathcal{A}=-\int_0^L\frac{\partial\mathcal{V}(z)}{\partial t_i}dz-\int_0^L dz\int_0^z d\zeta\left[\frac{\partial^2\mathcal{V}(z)}{\partial t_i\partial \mathbf{p}_i}\frac{\partial\mathcal{V}(\zeta)}{\partial \mathbf{r}_i}-\frac{\partial^2\mathcal{V}(z)}{\partial t_i\partial \mathbf{r}_i}\frac{\partial\mathcal{V}(\zeta)}{\partial \mathbf{p}_i}-\frac{\partial^2\mathcal{V}(z)}{\partial t_i\partial \mathcal{E}_i}\frac{\partial\mathcal{V}(\zeta)}{\partial t_i}+\frac{\partial^2\mathcal{V}(z)}{\partial t_i^2}\frac{\partial\mathcal{V}(\zeta)}{\partial \mathcal{E}_i}\right]. \tag{A12}$$

Upon averaging over the period of oscillations (equivalently over the injection phase of electrons) the terms which compose a full derivative over time give zero. Other terms are

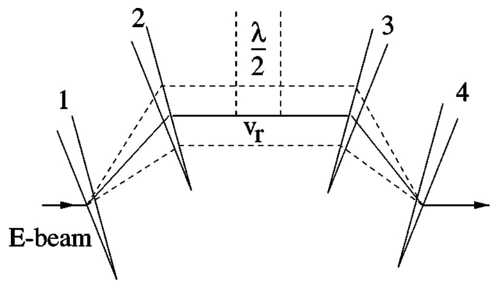

FIG. 8. The optical analog of the drift region.

$$\langle \mathcal{A} \rangle = -\int_0^L dz \int_0^z d\zeta \left[\frac{\partial^2 \mathcal{V}(z)}{\partial t_i \partial \mathbf{p}_i} \frac{\partial \mathcal{V}(\zeta)}{\partial \mathbf{r}_i} + \frac{\partial^2 \mathcal{V}(\zeta)}{\partial t_i \partial \mathbf{p}_i} \frac{\partial \mathcal{V}(z)}{\partial \mathbf{r}_i} - \frac{\partial^2 \mathcal{V}(z)}{\partial t_i \partial \mathcal{E}_i} \frac{\partial \mathcal{V}(\zeta)}{\partial t_i} - \frac{\partial^2 \mathcal{V}(\zeta)}{\partial t_i \partial \mathcal{E}_i} \frac{\partial \mathcal{V}(z)}{\partial t_i} \right]. \tag{A13}$$

Restoring the dynamical variables (A7)–(A10) and performing the integration over z, we arrive at our modified Madey's theorem where partial derivatives do not apply to $\Delta\phi$,

$$\langle \mathcal{A} \rangle \equiv \langle \mathcal{E}_2 \rangle = \left\langle \mathcal{E}_1(\mathcal{E}_i, \mathbf{p}_i, \Delta\phi) \frac{\partial \mathcal{E}_1(\mathcal{E}_i, \mathbf{p}_i, \Delta\phi)}{\partial \mathcal{E}_i} + \mathbf{p}_1(\mathcal{E}_i, \mathbf{p}_i, \Delta\phi) \frac{\partial \mathcal{E}_1(\mathcal{E}_i, \mathbf{p}_i, \Delta\phi)}{\partial \mathbf{p}_i} \right\rangle. \tag{A14}$$

This general theorem gives Eq. (45) for the case of the FEL. This can be shown either by expressing the detuning Ω from Eq. (17) via the energy $\mathcal{E}$ and the momentum $\mathbf{p}$, or by directly applying Eq. (A14) to the pendulum Hamiltonian (23).

APPENDIX B: IMPLEMENTATION OF THE PHASE DELAY IN THE DRIFT REGION

In the previous sections we have shown that there is no contradiction to the general theorems if we have a drift region providing the phase shift in the form (47). The aim of this appendix is to show that such a drift region could be designed in principle. The recent progress in electron microscopy [15] gives us a clue how we can use optical elements as analogs for electron optics [15]. Our efforts are devoted to the demonstration of the ''optical analog'' of the drift region as a proof of principle for the existence of such a type of drift region.

Below, we present the design by means of electron ''prisms,'' and electron ''plates'' which are acting almost in the same way as prisms and plates in optics. An electronic prism deflects electrons by some angle which depends on the velocity of the electron (which is an effect similar to dispersion in optics). An electron plate is a region where electrons are delayed for some additional time. Both prisms and plates can be designed by means of some region in space with magnetic fields in a way analogous to an electron lens.

In Fig. 8, we present a scheme of the drift region. This can be divided into three parts: a splitter, a kink region, and a collector. The splitter (a combination of prisms 1 and 2) is a part of the drift region which sorts electrons according to their detunings and arranges their motion in the next stage of the drift region, the so-called kink region. The electron beam leaving the first wiggler enters the splitter. Prism 1 deflects it, also providing, due to dispersion, spatial separation of the electrons according to the v_z component of their velocity. Then the electrons pass through prism 2, their velocities become parallel, and the phase shift of any electron will depend on the path along which the electron passed inside the splitter.

The second stage (the kink region) is a part producing the largest possible phase shift by letting electrons travel through this space. In addition, it adds π to the phase shift in the case of a negative detuning. The relative phase shift, in comparison with the resonant electrons, depends on the distance between the prisms and the angle between them.

The last part (a combination of prisms 3 and 4) simply collects the electrons in the beam in order to enter the second wiggler. The collector has the action reverse to that of a splitter.

The phase shift $\Delta\phi$ created by the drift region is given by

$$\Delta\phi = \nu \left(\frac{L_e}{v_e} - \frac{L_\nu}{c} \right), \tag{B1}$$

where L_e and L_ν are path lengths being passed by electrons and light, respectively, v_e is the velocity of an electron, and ν is the frequency of the light beam. Now it is easy to check that this drift region produces the phase shift we need for FELWI. The path length traveled by the electron is

$$L_e = L_e^o + \delta L, \tag{B2}$$

where

$$\delta L = 2R\,\delta\alpha_e \left(\frac{1-\cos\,\alpha_r}{\cos^2\alpha_r(\tan\,\theta_p + \tan\,\alpha_r)} - \tan\,\alpha_r \right) \tag{B3}$$

is the path difference for an electron deflected by the angle $\delta\alpha_e$ from the trajectory of resonant electron, R is the path distance of resonant electron passing between prism 1 and prism 2, α_e is the deflection angle of the electron, and θ_p is the angle between prism 2 and the axis of the system. For a small deviation from the resonant velocity v_r we have

$$\delta\alpha_e = \frac{\partial \alpha_e}{\partial v_e}(v_e - v_r), \qquad \delta L = \mathcal{L}(v_e - v_r) \sim \Omega, \tag{B4}$$

and finally the phase shift has the form

$$\Delta\phi = \nu \left(\frac{L_e^o}{v_r} - \frac{L_\nu}{c} \right) + \nu \left(\mathcal{L} - \frac{L_e^o}{v_r} \right) \frac{v_e - v_r}{v_r}. \tag{B5}$$

Adjusting some parameters of the drift region (for example L_e^o, θ, $\partial\alpha_e/\partial v_e$) we can transform the phase shift into the form we need for FELWI, namely,

$$\Delta\phi = \Delta\phi_0 - L_W \Omega. \tag{B6}$$

The possibility to do this meets the condition

$$\mathcal{L} < \frac{L_e^o}{v_r}. \tag{B7}$$

At this stage we have a phase-shift derivative over the detuning as we wanted to, and the last task is to create a kink-type phase shift (47). This can be easily done by introducing a plate inside the kink region, so that the electrons having negative detuning may pass through this plate adding the π phase shift, but others avoiding it adding no shift (therefore the name being ''kink region''). Thus we succeed in designing the right phase shift in its dependence on the electron detuning at the *exit* from the first wiggler; but for FELWI we need to design a drift region as a function of the detuning at the *entrance* of the first wiggler. To achieve this we need a kind of ''fate teller'' which should tell us what has been the detuning of the electron before the interaction inside the first wiggler.

As was shown in [11], this task can be performed by tilting the laser beam relative to the wiggler through an angle θ_W. This oblique light propagation gives rise to an electron motion in the (x,z) plane. Knowing that the trajectories in the (v_x,v_z) plane are straight lines with a slope $\gamma^2\theta_W$ relative to the v_z axis we can determine the entrance detuning of an electron by resolving this electron motion in the (v_x,v_z) plane. The splitter transforms this motion from the (v_x,v_z) plane into the (x,y) plane. The last problem therefore reduces to a proper adjustment of the plate inside the kink region. To solve it we should rotate the plate by the angle

$$\tan\Theta=\gamma^2\frac{\theta_p}{\partial\alpha_e/\partial v_e} \tag{B8}$$

around the z axis to add the phase π for negative initial detunings.

[1] *Physics of Quantum Electronics*, edited by S. F. Jacobs, H. S. Pilloff, M. Sargent III, and M. O. Scully (Addison-Wesley, Reading, MA, 1978), Vol. 5; *Physics of Quantum Electronics*, edited by S. F. Jacobs, G. T. Moore, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spetzer (Addison-Wesley, Reading, MA, 1979); Vol. 7; *Physics of Quantum Electronics*, edited by S. F. Jacobs, G. T. Moore, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spetzer (Addison-Wesley, Reading, MA, 1981), Vol. 8; *Physics of Quantum Electronics*, edited by S. F. Jacobs, G. T. Moore, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spetzer (Addison-Wesley, Reading, MA, 1981), Vol. 9; C. A. Brau, *Free-Electron Lasers* (Academic, Boston, 1990); G. Dattoli, A. Renieri, and A. Torre, *Lectures on the Free Electron Laser Theory and Related Topics* (World Scientific, London, 1993).

[2] J. M. J. Madey, Nuovo Cimento B **50**, 64 (1978).

[3] D. A. G. Deacon and Ming Xie, IEEE J. Quantum Electron. **QE-21**, 939 (1985); L. K. Grover and R. H. Pantell, *ibid.* **QE-21**, 944 (1985); P. Luchini and S. Solimeno, *ibid.* **QE-21**, 952 (1985); K.-J. Kim, Nucl. Instrum. Methods Phys. Res. A **318**, 489 (1992); N. M. Kroll, in *Free-Electron Generators of Coherent Radiation*, edited by S. F. Jacobs, H. S. Pilloff, M. Sargent III, M. O. Scully, and R. Spetzer, Vol. 8 of *Physics of Quantum Electronics* (Addison-Wesley, Reading, MA, 1982).

[4] J. Liouville, J. Math. **3**, 349 (1838).

[5] A. M. Sessler (private communication).

[6] O. A. Kocharovskaya and Ya. I. Khanin, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 581 (1988) [JETP Lett. **48**, 630 (1988)]; S. E. Harris, Phys. Rev. Lett. **62**, 1033 (1989); M. O. Scully, S.-Y. Zhu, and A. Gavrielides, *ibid.* **62**, 2813 (1989); A. S. Zibrov, M. D. Lukin, D. E. Nikonov, L. W. Hollberg, M. O. Scully, V. L. Velichansky, and H. G. Robinson, *ibid.* **75**, 1499 (1995); G. G. Padmabandu, G. R. Welch, I. N. Shubin, E. S. Fry, D. E. Nikonov, M. D. Lukin, and M. O. Scully, *ibid.* **76**, 2053 (1996).

[7] G. Kurizki, M. O. Scully, and C. Keitel, Phys. Rev. Lett. **70**, 1433 (1993); B. Sherman, G. Kurizki, D. E. Nikonov, and M. O. Scully, *ibid.* **75**, 4602 (1995); D. E. Nikonov, B. Sherman, G. Kurizki, and M. O. Scully, Opt. Commun. **123**, 363 (1996).

[8] P. Luchini and H. Motz, *Undulators and Free-Electron Lasers* (Clarendon Press, Oxford, 1990).

[9] A. Yariv, *Quantum Electronics* (Wiley, New York, 1989), Chap. 13.

[10] R. Bonifacio and M. O. Scully, Opt. Commun. **32**, 291 (1980).

[11] D. E. Nikonov, M. O. Scully, and G. Kurizki, Phys. Rev. E **45**, 6780 (1996).

[12] K.-J. Kim, Nucl. Instrum. Methods Phys. Res. A **393**, 234 (1997).

[13] Yu. V. Rostovtsev, D. E. Nikonov, and M. O. Scully (unpublished).

[14] D. A. G. Deacon, K. E. Robinson, J. M. J. Madey, C. Bazin, M. Billardon, P. Elleaume, Y. Farge, J. M. Ortega, Y. Petroff, and M. F. Velghe, Opt. Commun. **40**, 373 (1982).

[15] G. Dattoli and A. Torre, in *Dynamical Symmetries and Chaotic Behavior in Physics*, edited by G. Maino, L. Fronzoni, and M. Pettini (World Scientific, Singapore, 1991).